R_n Contains a Division Ring iff R Does

Ayman Badawi

INTRODUCTION. Let R be a ring with 1, and let R_n denote the complete matrix ring of all $n \times n$ matrices over R under the usual matrix addition and multiplication. Recall $A, B \in R_n$ are similar iff there exists $P \in R_n$ such that $A = PBP^{-1}$. If $A \in R_n$ is similar over R to a diagonal matrix, then A is called [**1**] diagonable over R. For $B \in R_n$, b_{ij} denotes the entry of B in the ith row and jth column.

In this note, we give an alternative proof of [**1**, Theorem 1] which is quite shorter than that in [**1**]. We would like to point out that our proof begins exactly like the original.

Theorem ([**1**, Theorem 1]). *Let R be a ring with* 1 *for which each idempotent matrix in R_n is diagonable over R. Then R contains a division ring if and only if R_n contains a division ring.*

Proof: If R contains a division ring, then clearly R_n contains a division ring. Assume R_n contains a division ring K. The division ring K has an identity—call it J—and by the hypothesis $PJP^{-1} = I$ a diagonal matrix for some invertible matrix $P \in R_n$. Since the conjugation of R_n by P induces a ring automorphism of R_n, $M = PKP^{-1}$ is a division ring of R_n and has I as the identity. Hence I is a nonzero idempotent of R_n. Let $S = \{A \in M\colon A \text{ is diagonal}\}$. Since $I \in S$, S is not empty. We leave it to the reader to verify that S is a division subring of M. Since $I \neq 0$, there exists $1 \leq j \leq n$ such that i_{jj} is a nonzero idempotent of R. Let $D = \{a_{jj}\colon A \in S\}$. Then D is a division ring of R with i_{jj} as the identity.

We end this note with some examples that satisfy the hypothesis of the Theorem and with one example where the hypothesis fails. Let R be a commutative ring with 1. Then R is called *ID* (basal) as in [**7**] ([**2**]) iff for every $n \geq 1$ the idempotents of R_n are diagonizable. Foster [**2**] has shown that if R is a principal ideal domain, then R is *ID*. Seshadri [**6**] has shown that if R is a principal ideal domain, then $R[x]$ is *ID*. In particular if F is a field, then $F[x, y]$ is *ID*. Steger [**7**] has shown that if R is an elementary division ring (i.e., for every $n \geq 1$ and $A \in R_n$ there exist invertible matrices P, Q in R_n such that PAQ is diagcnal) then R is *ID*. Also; Steger has shown that if R is π-regular ring (i.e., for every x in R there exists $n \geq 1$ and y in R (n and y depending on x) such that $x^n y x^n = x^n$) then R is *ID*. In particular for every $m \geq 1$ Z_m (i.e., Z/mZ) is *ID* (Foster has shown independently that Z_m is *ID*).

Finally, Theorem 3 in [**7**] states that if R is *ID*, then every invertible ideal of R is principal. Thus if R is a Dedekind domain which is not principal, then R is not *ID*. In particular, let $R = Z[\sqrt{-5}]$ (Z is the set of all integers). Then R is a Dedekind domain, see [**4**, EX. 37, P. 70]. But R is not a unique factorization domain, for example 21 does not have unique factorization in R. Thus R is not principal and therefore it is not *ID*.

REFERENCES

1. Jacob T. B. Beard, Jr. and Robert McConnel, Matrix fields over the integers modulo m, *Linear Algebra and Its Applications* 14, (1976), 95–105.
2. A. L. Foster, Maximal idempotent sets in a ring with unit, *Duke Math. J.* 13 (1946) 247–258.
3. L. Gilman and M. Henriksen, Some remarks about elementary divisor rings, *Tran. Amer. Math. Soc.* 82 (1956), 362–365.
4. Harry C. Hutchins, *Examples of Commutative Rings*, Harry C. Hutchings, 1981.
5. I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.* 66 (1949), 464–491.
6. C. S. Seshadri, Triviality of vector bundles over affine space K_2, *Proc. Nat. Acad. of Sci. USA* 44 (1958), 456–458.
7. A Steger, Diagonability of idempotent matrices, *Pac. J. Math.* 19 (1966), 535–542.

Dedicated to Prof. Nick Vaughan on his retirement.

Department. of Mathematics
University of North Texas
Denton, TX. 76203

A Further Simplification of Dixon's Proof of Cauchy's Integral Theorem

Peter A. Loeb

The modification in [**1**] of Dixon's proof of the Cauchy Integral Theorem and Formula is based on the proposition stated below. In this note we give a proof of that proposition which is more suitable for undergraduate students. In what follows, G will be an open set in the complex plane $\mathbf{C}$, and γ will be a closed rectifiable curve. We write $f \in H(G)$ if f is holomorphic, i.e. analytic, in G, and we use the notation $D(z, r)$ for the disk $\{w \in \mathbf{C}: |w - z| < r\}$. The trace of γ in $\mathbf{C}$ is denoted by $\{\gamma\}$; we say the curve γ is in G when $\{\gamma\} \subset G$.

Proposition. *If γ is a curve in G, then for any $z \in \{\gamma\}$ there is a closed curve σ in G with $z \notin \{\sigma\}$ such that $\int_\gamma f = \int_\sigma f$ for all $f \in H(G)$.*

Proof: We assume that there is a point $\zeta \neq z$ with $\zeta \in \{\gamma\}$; otherwise the result is trivial. Pick $r > 0$ so that $D(z, r) \subset G$ and $\zeta \notin D(z, r)$. We will assume that γ is given by $\gamma(t)$ for $t \in [0, 1]$ and $\gamma(0) = \gamma(1) = \zeta$. By the uniform continuity of the mapping γ, there is a natural number n such that if s, $t \in [0, 1]$ and $|t - s| < 1/n$, then $|\gamma(t) - \gamma(s)| < r$. Partition the interval $[0, 1]$ using the points $0 < 1/n < \cdots < (n-1)/n < 1$. Let $0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ be the set of partition points k/n such that $\gamma(k/n) \neq z$. If between adjacent points x_i and x_{i+1} there is a point of the form k/n or any other point t_0 with $\gamma(t_0) = z$, then the path $\gamma(t)$, $x_i \leqslant t \leqslant x_{i+1}$, is in the disk $D(z, r)$. In this case, we may replace the